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THE PRODUCT THEOREM FOR TOPOLOGICAL ENTROPY

BY
L. WAYNE GOODWYN

The purpose of this paper is to prove the following product theorem:

THEOREM 2. *Let X and Y be compact Hausdorff spaces and let $T: X \to X$ and $S: Y \to Y$ be continuous. Then*

$$h(T \times S) = h(T) + h(S),$$

where h denotes topological entropy [1], *and $T \times S: X \times Y \to X \times Y$ is defined as*

$$T \times S(x, y) = (Tx, Sy) \quad \textit{for } (x, y) \in X \times Y.$$

This theorem is stated in [1] without the assumption that X and Y be Hausdorff. However, the first half of the proof depends on the assertion that for open covers α of X and β of Y, $N(\alpha \times \beta) = N(\alpha) \cdot N(\beta)$. Without much difficulty one can construct examples where this equality does not hold, so that the proof in [1] yields only that $h(T \times S) \leqq h(T) + h(S)$.

1. **Introduction.** By a flow we mean a pair (X, T), where X is a compact Hausdorff space and $T: X \to X$ is a continuous map. Throughout the paper, (X, T) and (Y, S) will denote arbitrary flows. If T is a homeomorphism from X onto X, we say that (X, T) is a cascade.

In §2 we prove the product theorem for cascades, proving it first for subcascades of sequence cascades. In §3 we prove an inverse limit theorem which is used in §4 to generalize the product theorem. In §5 we give an example which shows that closed covers can, in general, yield larger entropy than open covers.

We shall use some notation and results from both [1] and [3]. We say that a cover β of X is a shrinking of a cover α if there is a function g from a subcollection α' of α onto β such that

$$g(A) \subset A \quad \text{for all } A \in \alpha'.$$

It is clear that in this case, the order of β is less than or equal to the order of α.

We say that a cover is nonoverlapping if the intersection of any two different members has no interior. Note that a shrinking of a nonoverlapping cover is nonoverlapping. If (X, T) is a cascade and if α is a finite closed nonoverlapping cover of X, then $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ is also a finite closed nonoverlapping cover for any positive

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integer n. Here, it is essential that T be a homeomorphism, and it is for this reason that we restrict our attention at first to cascades.

PROPOSITION 1. *Let α be a finite closed nonoverlapping cover of X. Then $N(\alpha)$ is the number of sets of α which have interior.*

Proof. Let β be the collection of sets in α which have interior, and let W be the union of the sets in α which do not have interior. Then

$$\bigcap_{A\in\beta}(X-A)\subset W.$$

But $\bigcap_{A\in\beta}(X-A)$ is open, and W has no interior, so $\bigcap_{A\in\beta}(X-A)$ must be empty, which means that β is a cover of X. On the other hand, each set of β is essential to α, for if $A\in\beta$ were not essential, then $A\subset\bigcup_{B\in\alpha;\,B\neq A}B$, so $A\subset\bigcup_{B\in\alpha;\,B\neq A}B\cap A$, contradicting the fact that A has interior. Hence β is a subcover of α of minimal cardinality.

COROLLARY 1. *Let α and β be finite closed nonoverlapping covers of X and Y respectively. Then $N(\alpha\times\beta)=N(\alpha)\cdot N(\beta)$. Hence, if (X, T) and (Y, S) are cascades, $h(\alpha\times\beta, T\times S)=h(\alpha, T)+h(\beta, S)$.*

PROPOSITION 2. *Every finite closed cover of X has a closed nonoverlapping shrinking.*

Proof. Let $\alpha=\{A_1,\dots,A_q\}$ be a closed cover. Define $B_i=A_i-\bigcup_{j<i}A_j^0$, $i=1,\dots,q$, where A_j^0 denotes the interior of A_j. Then for $i<k$, $B_i\cap B_k\subset A_i-A_i^0$, and so $B=\{B_1,\dots,B_q\}$ is nonoverlapping.

2. **The theorem for cascades.** We let I denote the interval $[0, 1]$ and we let Z denote the set of integers and Z^+ the set of nonnegative integers. For a positive integer m, we let C_m denote the set of all bisequences $(\dots x(-1), x(0), x(1),\dots)$ of points in I^m, i.e., $C_m=(I^m)^Z$. We let C_m have the product topology and we define $\sigma_m\colon C_m\to C_m$ by the rule

$$\sigma_m(x)(n) = x(n+1) \quad \text{for } x\in C_m,\ n\in Z.$$

We let $\pi_m\colon C_m\to I^m$ be the projection

$$\pi_m(x) = x(0) \quad \text{for } x\in C_m.$$

We shall use the following theorem from dimension theory:

THEOREM. *For every open cover of I^m there is a finite closed refinement of order $\leqq m+1$.*

If (X, T) is a cascade, then it is easily seen that, for positive integers p and n and a cover α of X,

$$\bigvee_{i=-p}^{p+n}T^{-i}\alpha\subset\bigvee_{j=0}^{n}T^{-j}\Big(\bigvee_{i=-p}^{p}T^{-i}\alpha\Big).$$

It follows by a straightforward argument that

$$h\left(\bigvee_{i=-p}^{p} T^{-i}\alpha, T\right) = h(\alpha, T).$$

LEMMA 1. *Let (X, T) and (Y, S) be subcascades of (C_m, σ_m). Then*

$$h(T)+h(S) \leqq h(T\times S)+2\log(m+1).$$

Proof. Let α be an open cover of C_m. Because of the definition of the topology for C_m, we can refine α to a *basic* open cover of the form

$$\alpha' = \bigvee_{i=-p}^{p} \sigma_m^{-i}\pi_m^{-1}(\beta_i)$$

where p is a positive integer and the β_i are open covers of I^m. Furthermore, taking $\beta = \bigvee_{i=-p}^{p} \beta_i$, we have

$$\bigvee_{i=-p}^{p} \sigma_m^{-i}\pi_m^{-1}(\beta) \succ \alpha.$$

Now by the above mentioned theorem in dimension theory, we can choose a finite closed cover γ of I^m of order $\leqq m+1$ which is a refinement of β. Letting $\beta_1 = \{\pi_m^{-1}(B) \cap X \mid B \in \beta\}$ and $\beta_2 = \{\pi_m^{-1}(B) \cap Y \mid B \in \beta\}$, we can apply Proposition 2 to obtain shrinkings δ_1 of β_1 and δ_2 of β_2 which are closed nonoverlapping covers of X and Y respectively. Now δ_1 and δ_2 each have order $\leqq m+1$, so $\delta_1\times\delta_2$ has order $\leqq (m+1)^2$, and it follows from Proposition 2 of [3] that

$$h(\delta_1\times\delta_2, T\times S) \leqq h(T\times S)+2\log(m+1).$$

Now using Corollary 1,

$$\begin{aligned} h_X(\alpha, T)+h_Y(\alpha, S) &\leqq h\left(\bigvee_{i=-p}^{p} T^{-i}\beta_1, T\right)+h\left(\bigvee_{i=-p}^{p} S^{-i}\beta_2, S\right) \\ &= h(\beta_1, T)+h(\beta_2, S) \leqq h(\delta_1, T)+h(\delta_2, S) \\ &= h(\delta_1\times\delta_2, T\times S) \leqq h(T\times S)+2\log(m+1). \end{aligned}$$

Now since α was an arbitrary open cover, the lemma is proved.

COROLLARY 2. *If (X, T) and (Y, S) are subcascades of (C_m, σ_m), then*

$$h(T)+h(S) = h(T\times S).$$

Proof. Let n be a positive integer. Then (X, T^n) and (Y, S^n) are subcascades of (C_m, σ_m^n). But it can be shown, as in Proposition 3 of [3], that (C_m, σ_m^n) is isomorphic to (C_{mn}, σ_{mn}), so that from the lemma, $h(T^n)+h(S^n) \leqq h(T^n\times S^n)+2\log(mn+1)$. Now, using Theorem 2 of [1], we have

$$\begin{aligned} h(T)+h(S)-h(T\times S) &= (1/n)(h(T^n)+h(S^n)-h((T\times S)^n)) \\ &\leqq (1/n)\cdot 2\log(mn+1). \end{aligned}$$

We now let n tend to infinity and observe that $(1/n)\cdot 2\log(mn+1)$ tends to zero, giving $h(T)+h(S)\leqq h(T\times S)$. Since the reverse inequality was proved in [1], we are done.

If (X, T) is a cascade and ϕ is a homomorphism from (X, T) into some (C_m, σ_m), we say that ϕ is a representation of (X, T), and we let T_ϕ denote the restriction of σ_m to $\phi(X)$. We let $R'(X, T)$ denote the set of all representations of (X, T). The following proposition is completely analogous to Theorem 2 of [3], and the proof will be omitted.

PROPOSITION 3. $h(T)=\sup\{h(T_\phi) \mid \phi\in R'(X, T)\}$.

LEMMA 2. *If (X, T) and (Y, S) are cascades, then $h(T\times S)=h(T)+h(S)$.*

Proof. Let $\varepsilon>0$. Then by the above proposition, we can choose $\phi\in R'(X, T)$ and $\psi\in R'(Y, S)$ such that $h(T)\leqq h(T_\phi)+\varepsilon/2$ and $h(T)\leqq h(S_\psi)+\varepsilon/2$. It follows from Corollary 2 that

$$h(T)+h(S) \leqq h(T_\phi)+h(S_\psi)+\varepsilon = h(T_\phi\times S_\psi)+\varepsilon.$$

However, the cascade $(\phi(X)\times\phi(Y), T_\phi\times S_\psi)$ is a homomorphic image of the cascade $(X\times Y, T\times S)$, so $h(T)+h(S)\leqq h(T\times S)+\varepsilon$. Hence, $h(T)+h(S)\leqq h(T\times S)$, and since the reverse inequality is known, the lemma is proved.

3. **A theorem on inverse limits.** In this section we assume that there is given a directed set $\mathscr{I}$, a system of flows $\{(X_i, T_i)\}_{i\in\mathscr{I}}$, and a consistent collection of homomorphisms $\lambda_{ij}\colon (X_i, T_i)\to(X_j, T_j)$ for $i, j\in\mathscr{I}$, $i\geqq j$.

The inverse limit of this system is defined to be the flow (X, T), where

$$X = \left\{x = (x_i)_{i\in\mathscr{I}}\in\prod_{i=\mathscr{I}} X_i \;\middle|\; \lambda_{jk}(x_j) = x_k \text{ for } j, k\in\mathscr{I}, j\geqq k\right\},$$

and $T\colon X\to X$ is given by

$$(T(x))_i = T_i(x_i) \quad \text{for } x\in X,\, i\in\mathscr{I}.$$

X is given the topology induced from the product topology on $\prod_{i\in\mathscr{I}} X_i$. We let $\lambda_i\colon X\to X_i$ be the coordinate projections restricted to X.

THEOREM 1. $h(T)\leqq\sup_{i\in\mathscr{I}} h(T_i)$, *and if the bonding maps λ_{ij} are all surjective, then* $h(T)=\lim_{i\in\mathscr{I}} h(T_i)$.

Before proving Theorem 1, we note that the requirement that the bonding maps be surjective is necessary for the equality $h(T)=\lim_{i\in\mathscr{I}} h(T_i)$. For if we let $X_i=\{x\in C_1 \mid x(n)\leqq 1/i \text{ for all } n\in Z\}$, and $T_i=\sigma_1|_{X_i}$, for $i\in Z^+$, and if we let the λ_{ij} be the inclusion maps, then it can be easily seen that the inverse limit flow (X, T) is the trivial one-point flow. On the other hand, it follows from work in [1] that $h(T_i)=\infty$ for $i\in Z^+$.

To prove Theorem 1, we need a lemma about inverse limits of compact Hausdorff spaces which has nothing to do with flows.

LEMMA 3. *For each open cover α of X, there is an index $k \in \mathscr{I}$ and an open cover β of X_k such that $\lambda_k^{-1}(\beta) \succ \alpha$.*

Proof. Because of the compactness of X and the definition of the product topology, we can choose a basic refinement of α of the form:

$$\gamma = \lambda_{i_1}^{-1}(\gamma_1) \vee \lambda_{i_2}^{-1}(\gamma_2) \vee \cdots \vee \lambda_{i_q}^{-1}(\gamma_q),$$

where $\{i_1, \ldots, i_q\}$ is a finite subset of $\mathscr{I}$, and $\gamma_1, \ldots, \gamma_q$ are open covers of $X_{i_1}, \ldots, X_{i_q}$ respectively. We now let $k = \max\{i_1, \ldots, i_q\}$ and note that because of the consistency of the λ_{ij}, we have

$$\lambda_{i_r}^{-1}(\gamma_r) = \lambda_k^{-1}(\lambda_{ki_r}^{-1}(\gamma_r)) \quad \text{for } r = 1, \ldots, q.$$

Now letting $\beta = \bigvee_{r=1}^{q} \lambda_{ki_r}^{-1}(\gamma_r)$, we have $\lambda_k^{-1}(\beta) = \gamma \succ \alpha$, and we are done.

Proof of Theorem 1. For each open cover α of X, we let k and β be as in the lemma. It follows from Property 7 of [1], that $h(\lambda_k^{-1}(\beta), T) \leqq h(\beta, T_k)$. Hence,

$$h(\alpha, T) \leqq h(\lambda_k^{-1}(\beta), T) \leqq h(\beta, T_k) \leqq h(T_k) \leqq \sup_{i \in \mathscr{I}} h(T_i).$$

Since α was arbitrary, we have $h(T) \leqq \sup_{i \in \mathscr{I}} h(T_i)$. Now if the λ_{ij} are assumed to be surjective, then the λ_i are also surjective, and it follows that $h(T) \geqq h(T_i)$ for all $i \in \mathscr{I}$. It also follows that, for $i \geqq j$, $h(T_i) \geqq h(T_j)$, so that $h(T) = \lim_{i \in \mathscr{I}} h(T_i)$.

4. The general product theorem. In this section we generalize Lemma 2, using a method introduced in [2] for constructing a cascade from a given flow.

If (X, T) is a flow, we let (X^*, T^*) be the inverse limit of $\{(X_i, T_i)\}_{i \in Z^+}$, where each $(X_i, T_i) = (X, T)$, and the bonding maps are given by $\lambda_{ij} = T^{i-j}$, for $i \geqq j$. Theorem 1 yields $h(T^*) \leqq h(T)$, and in fact, we shall show $h(T^*) = h(T)$. We note that (X^*, T^*) is a cascade; for the inverse of T^* is the shift, σ, defined by $(\sigma(x))_i = x_{i+1}$, for $x \in X^*$, $i \in Z^+$.

We define

$$\tilde{X} = \bigcap_{n \in Z^+} T^n(X) \quad \text{and} \quad \tilde{T} = T|\tilde{X}.$$

It is evident that $\tilde{T}(\tilde{X}) \subset \tilde{X}$, and we claim that, in fact, $\tilde{T}(\tilde{X}) = \tilde{X}$. For let $x \in \tilde{X}$. Then for each $n \in Z^+$, $T^{-1}(x) \cap T^n(X) \neq \varnothing$, and it follows from the compactness of $T^{-1}(x)$ that

$$T^{-1}(x) \cap \tilde{X} = T^{-1}(x) \cap \bigcap_{n=1}^{\infty} T^n(X) \neq \varnothing,$$

so that $\tilde{T}$ maps $\tilde{X}$ onto X.

PROPOSITION 4. *If U is an open set containing $\tilde{X}$, then there is some $k \in Z^+$ such that $T^k(X) \subset U$.*

Proof. This follows from the compactness of X, and the fact that

$$\{X - T^n(X) \mid n \in Z\} \cup \{U\}$$

is an open cover of X.

PROPOSITION 5. $h(\tilde{T})=h(T)$.

Proof. Let α be an open cover of X and let $\varepsilon>0$. From the definition of $h_{\tilde{X}}(\alpha, \tilde{T})$, we can choose $m \in Z^+$ so large that

$$\frac{1}{m}\log N_{\tilde{X}}\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha\right) < h_{\tilde{X}}(\alpha, \tilde{T})+\varepsilon.$$

Let β be a subcover of $\bigvee_{i=0}^{m-1} T^{-i}\alpha$ of minimal cardinality relative to $\tilde{X}$. By Proposition 4, we can choose a k so large that $T^{-k}(\beta)$ covers X. Hence, using Property 4 and Property 7 of [1], we have, for any $n \in Z^+$,

$$\begin{aligned} N\left(\bigvee_{i=0}^{mn-1} T^{-i}\alpha\right) &\leqq N\left(\bigvee_{i=0}^{mn+k-1} T^{-i}\alpha\right) \\ &= N\left(\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right) \vee T^{-k}\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha\right) \vee T^{-m}T^{-k}\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha\right)\right. \\ &\qquad \left. \vee \cdots \vee T^{-m(n+1)}T^{-k}\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha\right)\right) \\ &\leqq N\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right)\cdot\left(N\left(T^{-k}\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha\right)\right)\right)^n \leqq N\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right)\cdot(N_{\tilde{X}}(\beta))^n. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{mn}\log N\left(\bigvee_{i=0}^{mn-1} T^{-i}\alpha\right) &\leqq \frac{1}{mn}\log N\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right)+\frac{1}{m}\log N_{\tilde{X}}(\beta) \\ &< \frac{1}{mn}\log N\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right)+h_{\tilde{X}}(\alpha, \tilde{T})+\varepsilon. \end{aligned}$$

Now, letting n tend to infinity, we note that $(1/mn)\log N(\bigvee_{i=0}^{k-1} T^{-i}\alpha)$ tends to zero, so we have $h(\alpha, T)\leqq h_{\tilde{X}}(\alpha, \tilde{T})+\varepsilon\leqq h(\tilde{T})+\varepsilon$. Since ε and α were arbitrary, we have $h(T)\leqq h(\tilde{T})$. But the reverse inequality follows from Theorem 4 of [1], so the proposition is proved.

PROPOSITION 6. $h(T^*)=h(T)$.

Proof. We simply note that $(\tilde{X}^*, \tilde{T}^*)=(X^*, T^*)$ so that, by Proposition 5 and the second part of Theorem 1, we have $h(T)=h(\tilde{T})=h(\tilde{T}^*)=h(T^*)$.

THEOREM 2. $h(T\times S)=h(T)+h(S)$.

Proof. We first note that $(X^*\times Y^*, T^*\times S^*)$ is canonically isomorphic to $((X\times Y)^*, (T\times S)^*)$. Hence, by Proposition 6 and Lemma 2,

$$h(T\times S) = h((T\times S)^*) = h(T^*\times S^*) = h(T^*)+h(S^*) = h(T)+h(S).$$

We can generalize even further:

THEOREM 3. *Let $\{(X_k, T_k)\}_{k\in\mathscr{K}}$ be an arbitrary collection of flows. Let $X=\prod_{k\in\mathscr{K}} X_k$, and let $T\colon X\to X$ be defined by*

$$(T(x))_k = T_k(x_k), \qquad x\in X, k\in\mathscr{K}.$$

Then $h(T)=\sum_{k\in\mathscr{K}} h(T_k)$.

Proof. We let $\mathscr{I}$ be the collection of all finite subsets of $\mathscr{K}$, ordered by inclusion. We let $(Y_i, S_i)=\prod_{k\in i}(X_k, T_k)$, and we let λ_{ij} be the natural projections. It is straightforward to show that (X, T) is isomorphic to the inverse limit of the system $\{(Y_i, S_i)\}_{i\in\mathscr{I}}$. We now note that, by Theorem 2, $h(S_i)=\sum_{k\in i} h(T_k)$, and so, by the second part of Theorem 1,

$$h(T) = \lim_{i\in\mathscr{I}} h(S_i) = \lim_{i\in\mathscr{I}} \sum_{k\in i} h(T_k) = \sum_{k\in\mathscr{K}} h(T_k).$$

5. **An example.** In this section we construct a cascade which has topological entropy equal to zero, and which has a finite closed cover having entropy equal to log 2.

The cascade will be a subcascade of the sequence cascade (C_1, σ_1) defined in §2. For each positive integer m, we define X_m to be the set of all bisequences $x=(\cdots x(-1), x(0), x(1),\cdots)\in C_1$ satisfying the requirement that there are no more than m integers i such that $|x(i)-\frac{1}{2}|>1/m$. It can be checked that X_m is closed and σ_1-invariant. We now define

$$X = \bigcap_{m=1}^{\infty} X_m, \qquad T = \sigma_1|_X \quad\text{and}\quad \pi = \pi_1|_X.$$

To show that $h(T)=0$, it will suffice to show that $h(\pi^{-1}(\alpha), T)=0$ for open covers α of I, because every open cover of X has a refinement of the form $\bigvee_{i=-q}^{q} T^{-i}\pi^{-1}\alpha$. Let us write $\alpha=\{V_1,\ldots, V_t\}$, and $U_i=\pi^{-1}(V_i)$ for $i=1,\ldots, t$. We can choose $V_r\in\alpha$ such that $\frac{1}{2}\in V_r$, and we can choose m so large that the interval $[\frac{1}{2}-1/m, \frac{1}{2}+1/m]$ is contained in V_r. For a fixed integer n larger than m, we let $\mathscr{A}$ be the set of all subsets of $\{0, 1,\ldots, n-1\}$ with exactly m members, so that $\mathscr{A}$ contains $n!/m!(n-m)!$ sets. For $A\in\mathscr{A}$, we let A' denote the complement of A in $\{0, 1,\ldots, n-1\}$. For a function c from a set $A\in\mathscr{A}$ into the set $\{1,\ldots, t\}$, we define

$$W_c = \Big(\bigcap_{i\in A'} T^{-i}U_r\Big) \cap \Big(\bigcap_{i\in A} T^{-i}U_{c(i)}\Big),$$

which is a member of $\bigvee_{i=0}^{n-1} T^{-i}\pi^{-1}(\alpha)$. We let

$$\gamma = \{W_c \mid A\in\mathscr{A} \text{ and } c\colon A\to\{1,\ldots, t\}\},$$

and we observe that γ has no more than $(n!/m!(n-m)!)p^m$ members, hence no more than $n^m p^m$ members. Next, we claim that γ covers X. Let $x\in X$. Then $x\in X_m$, so there is a set $A\in\mathscr{A}$ such that $|x(i)-\frac{1}{2}|\leqq 1/m$ for all $i\in A'$. Thus, for $i\in A'$, $\pi(T^i x)=\sigma^i(x)(0)=x(i)\in V_r$, so that $T^i x\in U_r$, and hence $x\in\bigcap_{i\in A'} T^{-i}U_r$. Now, for $i\in A$, we choose $c(i)$ so that $T^i x\in U_{c(i)}$, and it follows that $x\in W_c$. This shows that γ covers X, so that we have a subcover of $\bigvee_{i=0}^{n-1} T^{-i}\pi^{-1}(\alpha)$ with no more than $n^m p^m$ members, and we have $N(\bigvee_{i=0}^{n-1} T^{-i}\pi^{-1}(\alpha))\leqq n^m p^m$. Hence,

$$\frac{1}{n}\log N\Big(\bigvee_{i=0}^{n-1} T^{-i}\pi^{-1}(\alpha)\Big) \leqq \frac{1}{n}\log p^m n^m,$$

and letting n tend to infinity, we obtain $h(\pi^{-1}(\alpha), T)=0$. Hence, the topological entropy of T is zero.

We now define $F_0=\{x \in X : x(0) \in [0, \frac{1}{2}]\}$ and $F_1=\{x \in X : x(0) \in [\frac{1}{2}, 1]\}$, and let $\beta=\{F_0, F_1\}$. We claim that $h(\beta, T)=\log 2$. Let n be a fixed positive integer. For each function b from $\{0, \ldots, n-1\}$ into $\{0, 1\}$, we let

$$G_b = F_{b(0)} \cap T^{-1}F_{b(1)} \cap \cdots \cap T^{-n+1}F_{n-1} \in \bigvee_{i=0}^{n-1} T^{-i}\beta$$

and we let x_b be the point of X defined by the rule

$$\begin{aligned} x_b &= \tfrac{1}{2}-(-1)^{b(i)}/n \quad \text{for } i = 0, 1, \ldots, n-1, \\ &= 0 \qquad\qquad\qquad\quad \text{for } i < 0 \text{ or } i \geqq n. \end{aligned}$$

Then it can be shown that $x_b \in G_b$ and that $x_b \notin G_{b'}$ whenever $b' \neq b$. Thus, for each $b: \{0, \ldots, n-1\} \to \{0, 1\}$, G_b is an essential member of the cover $\bigvee_{i=0}^{n-1} T^{-i}\beta$, so that $N(\bigvee_{i=0}^{n-1} T^{-i}\beta)=2^n$. Hence,

$$\frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\beta\right) = \log 2.$$

Since n was arbitrary, $h(\beta, T)=\log 2$.

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University of Kentucky,
Lexington, Kentucky 40506